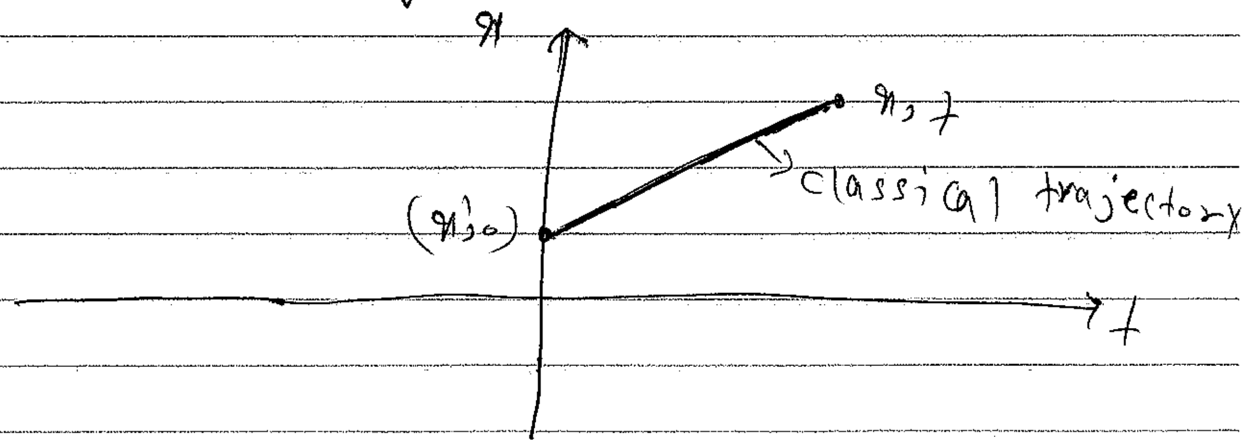
Lec 22: 10/19/2009

Classical Limit:

Now, the question is when we are at the classical limit. In the path integral formalism, a particle tries all paths going from one point to another. However, there are cases where the particle travels on a specific path clearly. How we are going to make sense of this? The classical limit is meaningful if it exists. So let us consider a case where there exists a classical path, namely free particle.



We want to see which paths are important. As pointed out, the classical path minimizes the action. Hence S changes slowly for paths that are sufficiently close to it. Those path for which $\Delta S = S - S_{cl}$ is $\lesssim \hbar\pi$ have a constructive interference with the classical path. Lets consider path whose maximum deviation from the classical path is Δx. We want to find how large Δx can be such that $\Delta S \lesssim \hbar\pi$.

$S = S_{cl} + \Delta S$

$$S = \int_0^t \frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2 d\tau = \int_0^t \frac{1}{2} m \left[\frac{d}{d\tau}(x_{cl} + \Delta x)\right]^2 d\tau$$

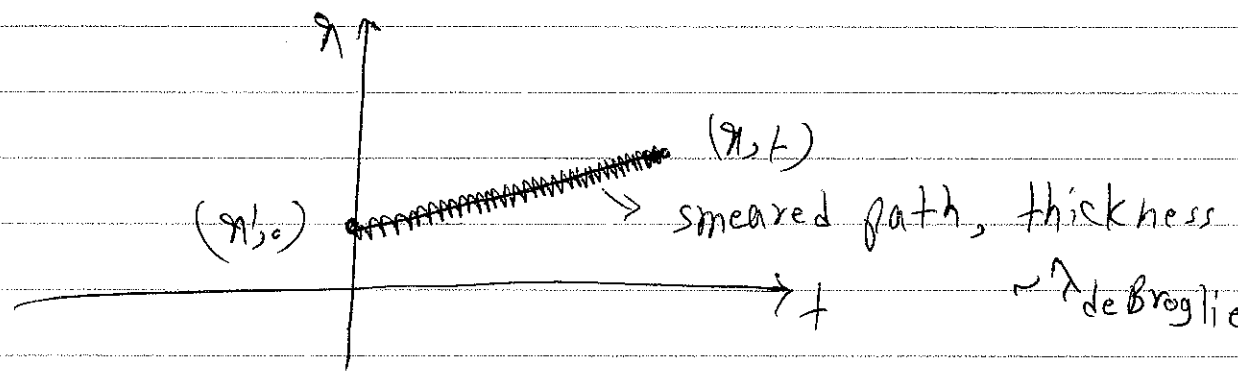
$$\simeq \int_0^t \frac{1}{2} m \left(\frac{dx_{cl}}{d\tau}\right)^2 d\tau + \frac{m\, dx_{cl}\, \Delta x}{dt}$$

Therefore we need:

$$\Delta S \simeq \frac{m\, dx_{cl}\, \Delta x}{dt} \lesssim \hbar\pi \Rightarrow \Delta x \lesssim \frac{\pi\hbar}{P_{cl}}$$

Note that $\lambda_{de\,Broglie} = \frac{h}{p}$. Therefore paths that are within a distance $\sim \lambda_{de\,Broglie}$ around the classical trajectory make the main contribution to the sum $\sum_{\text{all path}} e^{iS/\hbar}$.

This introduces a "quantum smearing" of the classical path:

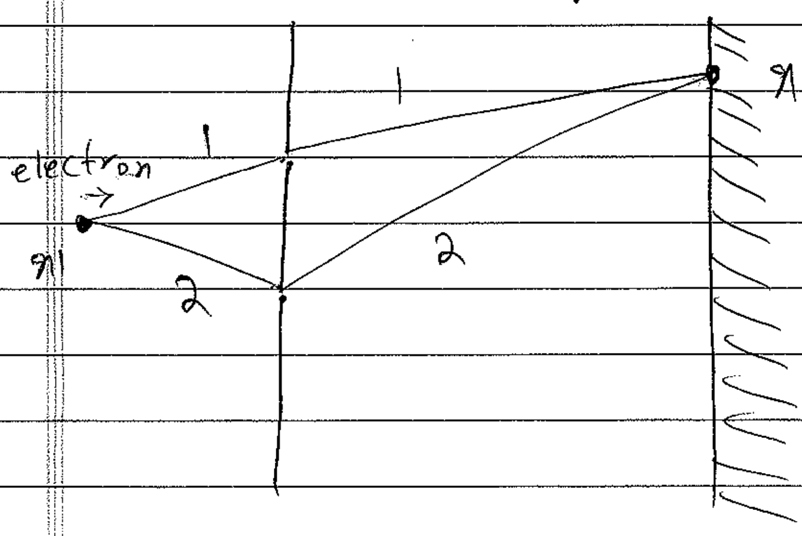


The point is how this smearing compares with the experimental uncertainty in determining the position of the particle. If the latter Δx_{exp} is larger than $\lambda_{de\,Broglie}$, then we won't notic the quantum smearing because the experimental uncertainty is already larger than that.

Then we are in the classical limit. We note that this is the same criterion that we found from the Heisenberg uncertainty principle. If $\Delta X_{exp} > \lambda_{deBroglie}$, we are in the classical limit. In the opposite situation $\Delta X_{exp} < \lambda_{deBroglie}$ we are in the domain of quantum mechanics.

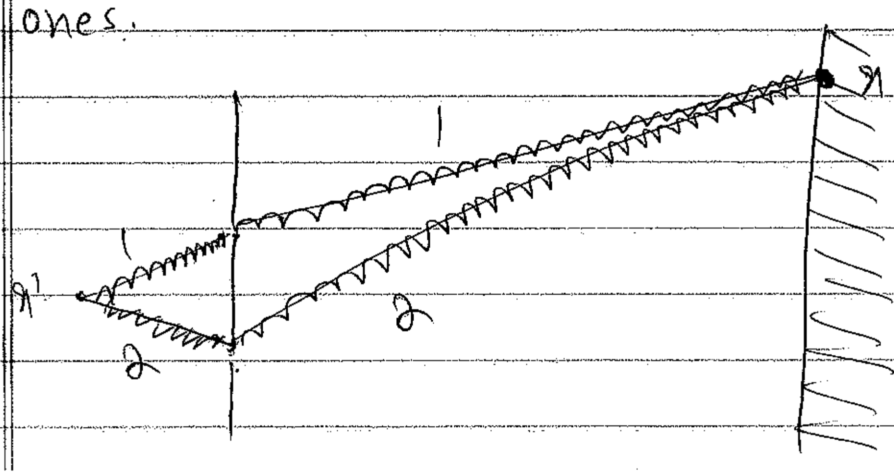
Multiple Classical Trajectories:

Next we consider cases where there exists more than one classical path. This happens, for example, in interference experiment:



Electrons are shot at a screen with two holes punched in it, pass through the holes and end up at a point on a second screen.

Classically, there are two possible paths, 1 and 2, and the particle travels along 1 <u>or</u> 2. On the other hand, it moves along all possible paths in quantum mechanics. The two classical trajectories are extrema of the action S. Hence, paths within $O(\lambda_{deBroglie})$ from these two paths make the major contribution. As a result, the two smeared paths 1 and 2 are the most important ones.



Thus, the probability to find the electron at a point x on the back screen follows,

$$|U(x,t;x')|^2 \approx |a\, e^{iS_1/\hbar} + b\, e^{iS_2/\hbar}|^2$$

Here a, b take the smearing into account. We have:

$$|U(x,t;x')|^2 \approx |a + b\, e^{\frac{i(S_2 - S_1)}{\hbar}}|^2$$

This explains why we observe a periodic interference pattern on the back screen. $S_2 - S_1$ increases (decreases) as we move upward (downward) on that screen. The factor $\exp\left[\frac{i(S_2-S_1)}{\hbar}\right]$ thus changes periodically and leads to an interference pattern for the intensity of electrons.

Note that:

$$S_2 - S_1 = {}^{2}\!\int_0^t L\,dt - {}^{1}\!\int_0^t L\,dt$$

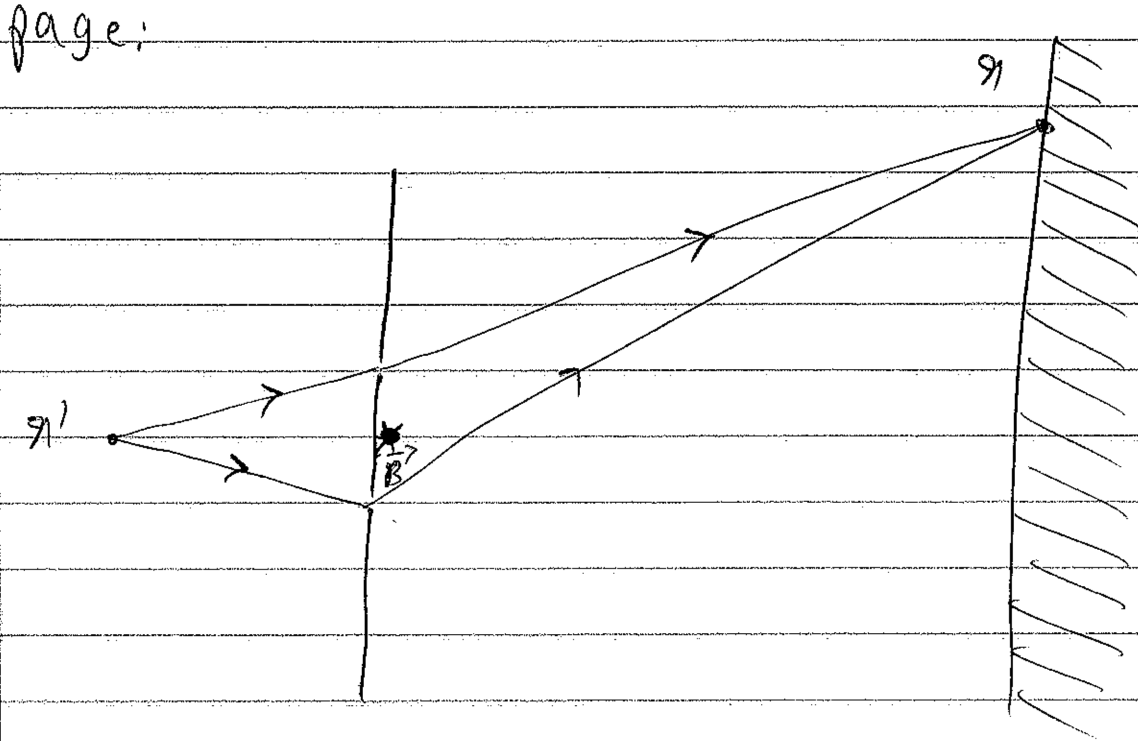
Here 1 and 2 denote paths 1, 2 respectively.

For a charged particle (here electrons), we have:

$$L = \frac{1}{2} m |\vec{v}|^2 + q\, \vec{v} \cdot \vec{A} - q\phi$$

$\vec{A}$ and ϕ are the vector and scalar potential respectively.

Now lets assume that we put a solenoid right behind the first screen. By turning on a current, it generates a magnetic field confined inside the solenoid that points to/from this page:



Note that in classical physics, the electron does not feel the magnetic field at all. What matters there is the force $\vec{F} = q\vec{v} \times \vec{B}$, which is zero everywhere outside the solenoid.

In quantum mechanics, however, what is important is $S_2 - S_1$:

$$S_2 - S_1 = {}^{2}\int_0^t \left(\frac{1}{2} m |\vec{v}|^2 + q\vec{v} \cdot \vec{A}\right) dt - {}^{1}\int_0^t \left(\frac{1}{2} m |\vec{v}|^2 + q\vec{v} \cdot \vec{A}\right) dt$$

Here $\phi = 0$ since there is no electric field. But,

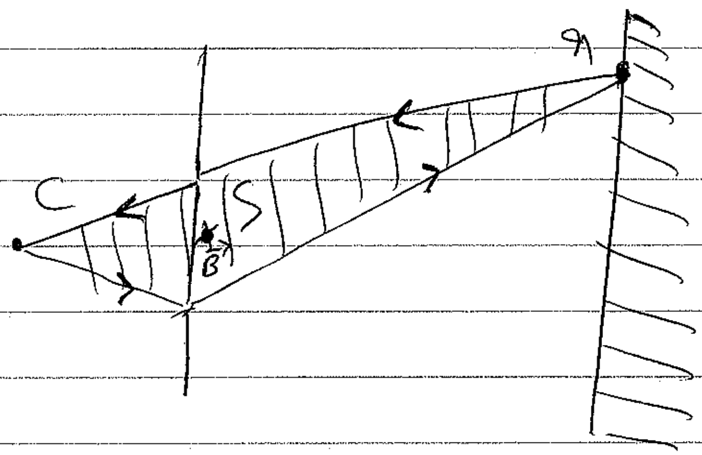
$$(\vec{v} \cdot \vec{A})\, dt = d\vec{r} \cdot \vec{A}$$

The part of $S_2 - S_1$ that depends on $\vec{A}$ is:

$$^{2}\int_{r'}^{r} q\vec{A} \cdot d\vec{r} - {}^{1}\int_{r'}^{r} q\vec{A} \cdot d\vec{r} = \oint_C q\vec{A} \cdot d\vec{r}$$

But:

$$\oint_C q\vec{A} \cdot d\vec{r} = \int_S q(\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int_S q\vec{B} \cdot d\vec{a}$$



If there is no current in the solenoid, $\int_S \vec{B}\cdot d\vec{a} = 0$.

While, for a non-zero current, we have $\int_S \vec{B}\cdot d\vec{a} \neq 0$

Hence, turning on the current changes $S_2 - S_1$ by a constant value. This results in a shift (upward or downward) in the interference pattern on the back screen.

This is the so called "Aharanov-Bohm" effect.

It is purely a quantum mechanical effect that does not exist in classical physics.